

Exact Solution to the homogeneous Maxwell Equations in the Field of a Gravitational Wave in Linearized Theory

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We present the exact solution to the linearized Maxwell equations in space-time slightly curved by a gravitational wave. We show that in general, even dealing with a first-order theory in the strength of the gravitational field, the solution can not be written as the sum of the flat space-time one and a weak perturbation due to the external field. Such an impossibility arises when either the frequency of the gravitational wave is too low or too high with respect to the one of the electromagnetic field. We also provide an application of the solution to the case of an electromagnetic field bounced between two parallel conducting planes.

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I. INTRODUCTION

The problem on the propagation of electromagnetic fields in a curved space-time has been widely studied all through the years. Einstein himself calculated the deflection of a light ray by the gravitational field of the Sun. This problem can be simply approached by assuming light rays as trajectories of a zero-mass particle, for which $ds^2 = 0$. Moreover the same approach has been used to calculate the response of an interferometer to an impinging gravitational wave [1].

A different approach to the problem is considering directly the Maxwell equations in curved space-time. Two kinds of phenomena can be described by this method: propagation of electromagnetic radiation in a non-radiative gravitational background [2–6] (see also Ref. [7] for an exhaustive treatise on the subject) and in the field of a gravitational wave. Not only are they interesting problems themselves but also have important experimental applications (see for instance [8]).

As for the propagation in a gravitational wave field, some authors have solved the problem by using the formalism of geometrical optics; therefore these results are valid under the assumption that the gravitational wave is slowly varying with respect to the electromagnetic field [9–11]. Others have solved the equations by splitting the electromagnetic tensor (or the four-vector potential) in a sum of two terms: the unperturbed solution and a correction term [12–20]. However, as it has been shown in Ref. [11], the splitting approach turns the problem of the propagation of a free electromagnetic field in a gravitational background into a different one, that is the generation of a field by a given current. On this ground many doubts could be raised as to the validity of this procedure (a detailed discussion may be found in Ref. [11]).

Therefore we see that a satisfactory solution that only assumes the smallness of the gravitational wave amplitude regardless of its shape or frequency is lacking. In this paper we provide exact solution to the problem by solving Maxwell equations for the 4-potential (De Rham equations) in the framework of the linearized general relativity (Sec. II). In other words the solution we have achieved is correct as long as the linearized De Rham equations (2.3) describe properly the physical problem. We express our result as an integral which is the curved space-time version of the usual Fourier integral. In this way any solution can be obtained by a proper choice of the Fourier-like coefficients. Afterwards we calculate the expressions of the electromagnetic tensor components (Sec. III). Finally Sec. IV is devoted to the application of our solution to a simple case in order to compare it to the one obtained through the “splitting” procedure.

Besides we find that the main effect of a low-frequency gravitational wave on an electromagnetic field bounced between two parallel conducting walls is ohmic power loss.

II. EXACT SOLUTION TO THE DE MAXWELL EQUATIONS

The homogeneous de Rham equations describe the propagation of electromagnetic field in an external curved space-time. Under the Lorentz condition, they could be written as:

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$$\begin{cases} A^{\mu;\alpha}_{;\alpha} = 0 \\ A^{\sigma}_{;\sigma} = 0 \end{cases} \quad (2.1)$$

(throughout this paper the notation and conventions are as in [21] except for indices p, r, s which take values 1 and 2). Let us assume that the curvature of space-time is due to a weak plane gravitational wave, propagating in the x^3 direction of a given *transverse traceless* (TT) reference frame [21]. In this case the metric tensor is:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^3 - x^0), \quad |h_{\mu\nu}| \ll 1, \quad (2.2)$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor with positive signature and $h_{\mu\nu}(x^3 - x^0)$ is the perturbation to the flat space-time metric. In the chosen reference frame the only non-vanishing components are $h_{11} = -h_{22} = h_+$ and $h_{12} = h_{21} = h_\times$. Neglecting second order terms in $h_{\mu\nu}$, Eqs. (2.1) become ($h^\mu_{\nu} = \eta^{\mu\alpha} h_{\alpha\nu}$):

$$\begin{cases} A^{\mu,\nu}_{;\nu} - h^{\alpha\beta} A^{\mu}_{;\alpha,\beta} + (h^\mu_{\alpha;\beta} + h^{\mu\beta}_{;\alpha} - h^\beta_{\alpha;\mu}) A^\alpha_{;\beta} = 0 \\ A^\sigma_{;\sigma} = 0 \end{cases} \quad (2.3)$$

Since the first equation of the above system for $\mu = 0$ is the same as $\mu = 3$, we can choose a particular Lorentz gauge in which $A^3 = A^0$. This greatly simplifies the problem, because this way the equations are partially decoupled [see Eqs. (2.5)–(2.8) later on]. Eqs. (2.3) are a set of differential equations with variable coefficients, which depend only on $x^3 - x^0$. This fact leads us to perform the following coordinate transformation:

$$\begin{cases} X^0 = (x^3 - x^0)/\sqrt{2} \\ X^1 = x^1 \\ X^2 = x^2 \\ X^3 = (x^3 + x^0)/\sqrt{2} \end{cases} \quad (2.4)$$

With this new set of variables, system (2.3) becomes (∂_μ means $\frac{\partial}{\partial X^\mu}$):

$$\hat{D}A^p + \partial_0 h^p_s \partial_3 A^s = 0 \quad (2.5)$$

$$\hat{D}A^3 - \partial_0 h^s_r \partial_s A^r = 0 \quad (2.6)$$

$$\hat{D}A^0 = 0 \quad (2.7)$$

$$A^0 = 0; \quad \partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3 = 0. \quad (2.8)$$

where

$$\hat{D} = (\partial_1 \partial_1 + \partial_2 \partial_2 + 2\partial_3 \partial_0 - h^{rs} \partial_r \partial_s),$$

$p, r, s = 1, 2$, and $A^\mu = \frac{\partial X^\mu}{\partial x^\nu} A^\nu$ are the new components of the 4-vector potential. We observe that in this new reference frame the electromagnetic gauge condition is the so-called Coulomb gauge [11]. Hereinafter only the real part of equations is to be retained. First of all we solve Eq. (2.5); through Fourier transform in X^j ($\mathbf{X} \rightarrow (X^1, X^2, X^3)$, $\boldsymbol{\lambda} \rightarrow (\lambda_1, \lambda_2, \lambda_3)$)

$$A^p(\mathbf{X}, X^0) = \int e^{i\lambda_k X^k} a^p(\boldsymbol{\lambda}, X^0) d^3 \boldsymbol{\lambda}, \quad (2.9)$$

one gets:

$$\left[2i\lambda_3 \frac{d}{dX^0} - (\lambda_1^2 + \lambda_2^2) + \lambda_r \lambda_s h^{rs} (\sqrt{2} X^0) \right] a^p + i\lambda_3 \frac{dh^p_r}{dX^0} a^r = 0. \quad (2.10)$$

Setting:

$$a^p = \gamma(X^0; \boldsymbol{\lambda}) B^p(\sqrt{2} X^0; \boldsymbol{\lambda}) \quad (2.11)$$

with

$$\gamma(X^0; \boldsymbol{\lambda}) = \exp \left[\frac{(\lambda_1^2 + \lambda_2^2) X^0 - \lambda_r \lambda_s h^{(-1)rs}(\sqrt{2} X^0)}{2i\lambda_3} \right], \quad (2.12)$$

($h^{(-1)r s}$ stands for any primitive of $h^{r s}$) Eq. (2.10) gives:

$$\frac{dB^p}{dX^0} + \frac{1}{2} \frac{dh^p_q}{dX^0} B^q = 0. \quad (2.13)$$

The solution to the above equation could be expressed as a function series:

$$\begin{aligned} B^r(w; \boldsymbol{\lambda}) &= \left[\delta^r_p + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \aleph^r_p(n; w) \right] b^p(\boldsymbol{\lambda}) \\ \aleph^r_p(n; w) &= \int \frac{dh^r_q}{d\psi} \aleph^q_p(n-1; w) dw \\ \aleph^r_p(1; w) &= \int \frac{dh^r_p}{dw} dw \end{aligned} \quad (2.14)$$

where $b^p(\boldsymbol{\lambda})$ depends only on $\boldsymbol{\lambda}$. As one can easily see, $\aleph^r_p(n; w)$ are of same order as h^n , where h is the order of magnitude of the gravitational wave amplitude. A sufficient condition for the series to converge is that $\aleph^r_p(n; w) \leq 1$; this is always satisfied under the assumption of linearized theory, in which $h \ll 1$. Therefore, through this approximation, we can keep only the first term in the series (2.14). In this case

$$B^r(\sqrt{2} X^0; \boldsymbol{\lambda}) = b^r(\boldsymbol{\lambda}) - \frac{1}{2} h^r_p(\sqrt{2} X^0) b^p(\boldsymbol{\lambda}). \quad (2.15)$$

As for the \mathcal{A}^3 component, the Coulomb gauge [Eq. (2.8)] gives:

$$a^3 = -\frac{\lambda_1 a^1 + \lambda_2 a^2}{\lambda_3} \quad (2.16)$$

which automatically fulfills Eq. (2.6).

Having solved the system of equations (2.5)–(2.8) in the X^μ reference frame [Eqs. (2.4)], we proceed to write the solution in the x^α coordinates. In the case of a flat space-time, i.e. $h_{\mu\nu} = 0$, k_j would be the Fourier variables conjugated to x^j coordinates. In this way we have that $\lambda_3 = (k_3 \pm k)/\sqrt{2}$ ($k = \sqrt{k_1^2 + k_2^2 + k_3^2}$). One gets ($\mathbf{k} \rightarrow (k_1, k_2, k_3)$):

$$A^p = \int B^p(x^3 - x^0; \mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{x}} \left[\beta^+(\mathbf{k}) e^{i(k t + \psi^+)} + \beta^-(\mathbf{k}) e^{-i(k t - \psi^-)} \right] d^3 \mathbf{k} \quad (2.17)$$

$$A^3 = A^0 = \int k_s B^s(x^3 - x^0; \mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{x}} \left[-\frac{\beta^+(\mathbf{k})}{k + k_3} e^{i(k t + \psi^+)} + \frac{\beta^-(\mathbf{k})}{k - k_3} e^{-i(k t - \psi^-)} \right] d^3 \mathbf{k} \quad (2.18)$$

where

$$\psi^\pm = \pm \frac{k_r k_s h^{(-1)r s} (x^3 - x^0)}{2 (k \pm k_3)} \quad (2.19)$$

and $\beta^\pm(\mathbf{k})$ are generic functions whose actual form is determined by assigning suitable initial or boundary conditions. The solution given in Eqs. (2.17) and (2.18) only assumes the smallness of $h_{\mu\nu}$ and does not require any further hypothesis about its rate of variation. From Eq. (2.19) it follows that:

$$\psi \sim \frac{k}{\chi} h$$

where χ is the order of magnitude for a typical frequency of gravitational wave. As no assumption was made over χ , ψ is not necessarily a first order term in h . Therefore, in the general case, it is not possible to expand the exponential terms in the solution to the lowest order in ψ , opposite to the case of the function series (2.14), in which there is no term proportional to $1/\chi$.

If the gravitational wave is linearly polarized then:

$$h_+(x^3 - x^0) = H_+ f(x^3 - x^0), \quad h_\times(x^3 - x^0) = H_\times f(x^3 - x^0) \quad (2.20)$$

where H_+ and H_\times are constants. We can set:

$$H_+ = H \cos 2\alpha, \quad H_\times = H \sin 2\alpha. \quad (2.21)$$

In this case the solution to Eq. (2.13) greatly simplifies. Actually it is possible to express it as a function rather than an infinite series of functions [see Eq. (2.14)]. Using standard methods the solution can be written:

$$B^p = b^p e^{-\frac{\lambda}{2} f(x^3 - x^0)} \quad (2.22)$$

where b^p and λ are the eigenvectors and eigenvalues of the following system:

$$H^p{}_q b^q = \lambda b^p. \quad (2.23)$$

The eigenvalues are $\lambda = H$ and $\lambda = -H$ while the corresponding eigenvectors are:

$$\mathbf{b}_+ = (\cos \alpha, \sin \alpha) \quad \text{and} \quad \mathbf{b}_- = (\sin \alpha, -\cos \alpha) \quad (2.24)$$

respectively. Therefore the solutions read

$$B_+^p = b_+^p e^{-\frac{H}{2} f(u)} \quad \text{and} \quad B_-^p = b_-^p e^{+\frac{H}{2} f(u)}. \quad (2.25)$$

III. THE ELECTROMAGNETIC TENSOR

Having found the solution to the de Rham equations, it is straightforward to determine the components of the electromagnetic tensor by means of

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \quad (3.1)$$

We explicitly have:

$$F_{\mu\nu} = \int e^{i\mathbf{k}\cdot\mathbf{x}} \left[\beta^+(\mathbf{k}) f_{\mu\nu}^+ e^{i(kx^0 + \psi^+)} + \beta^-(\mathbf{k}) f_{\mu\nu}^- e^{-i(kx^0 - \psi^-)} \right] d^3k \quad (3.2)$$

where (dot means derivation with respect to $x^3 - x^0$):

$$f_{03}^\pm = -i k_s \left(b^s - \frac{1}{2} h^s{}_r b^r \right) \quad (3.3)$$

$$f_{0p}^\pm = i \left[\pm k \left(b^p + \frac{1}{2} h^p{}_r b^r \right) \mp b^p \frac{k_r k_s h^{rs}}{2(k \pm k_3)} \mp \frac{k_p k_s}{k \pm k_3} \left(b^s - \frac{1}{2} h^s{}_r b^r \right) \right] - \frac{1}{2} \dot{h}^p{}_r b^r \quad (3.4)$$

$$f_{12}^\pm = i \left[k_1 \left(b^2 + \frac{1}{2} h^2{}_r b^r \right) - k_2 \left(b^1 + \frac{1}{2} h^1{}_r b^r \right) \right] \quad (3.5)$$

$$f_{p3}^\pm = i \left[-k_3 \left(b^p + \frac{1}{2} h^p{}_r b^r \right) \mp b^p \frac{k_r k_s h^{rs}}{2(k \pm k_3)} \mp \frac{k_p k_s}{k \pm k_3} \left(b^s - \frac{1}{2} h^s{}_r b^r \right) \right] - \frac{1}{2} \dot{h}^p{}_r b^r \quad (3.6)$$

Maxwell equations in vacuum, $F^{\mu\nu}{}_{;\nu} = 0$, in the linear approximation of the chosen TT gauge reference frame [see Eq. (2.2)] read:

$$(\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\alpha} h^{\nu\beta} - \eta^{\nu\beta} h^{\mu\alpha}) F_{\alpha\beta,\nu} + h^{\mu\alpha}{}_{,\nu} \eta^{\nu\beta} F_{\alpha\beta} = 0, \quad (3.7)$$

where terms of the order of h^2 and χh^2 are negligible. Actually the electromagnetic field components [Eqs. (3.2)–(3.6)] satisfy Eq. (3.7) with the same degree of precision. The homogeneous Maxwell equations are identically fulfilled because of Eq. (3.1).

Therefore our solution solves the problem of the propagation of an electromagnetic field in a gravitational wave background with the only assumption of a small perturbation to a flat metric, thus generalizing the result of [11] to any gravitational wave frequency.

IV. APPLICATION

As an example let us consider the case of an electromagnetic field bounced between two conducting parallel walls a distance L apart. In the absence of gravitational waves, the physical features of the problem we are concerned with do not depend on the specific orientation of the apparatus. This is due to the isotropy of flat space-time. However this isotropy breaks down in the presence of a radiative gravitational field propagating along a given direction. Because of this anisotropy, electromagnetic waves propagating in opposite directions are in general affected in different ways [see Eqs. (2.19) and (3.2)]. A consequence of this circumstance is the impossibility of obtaining stationary solutions. Only if the propagation of the electromagnetic field lies on the plane perpendicular to the propagation of the gravitational wave, a stationary solution does exist. In fact, only in this case, the perturbation to the phase of the electromagnetic wave is the same for opposite directions [see Eqs. (2.19) and (3.2)]. The peculiarity of this plane also arises if we notice that two different points lying on it undergo the same gravitational perturbation.

In Subsec. A we consider the last case and compare our result with the already existing solutions obtained by splitting of the electromagnetic field [12,20] (see Sec. I). We find out that the latter approach lies in the framework of linearized theory only in the definite range of gravitational wave frequencies given by the relation:

$$k h \ll \chi \ll \frac{k}{h} \quad (4.1)$$

This is different from what was pointed out in Ref. [12], where such a condition was mandatorily needed for the linearized theory to be valid. Indeed this is merely the condition for the validity of the splitting procedure. We think that this circumstance is due to the expansion in power of the gravitational wave amplitude of the electromagnetic phase, expansion that is needed by the splitting method. This kind of procedure (expansion of a phase) is not in general correct, as it is shown, for instance, by the long experience in perturbative methods in both classical and relativistic celestial mechanics (e.g. calculation of the advance of the perihelion of Mercury [22]).

In Subsec. B we show, in a particular case, how the non stationariness of the solution induces dissipative effects on the conducting planes.

A. Comparison with the splitting procedure

As a first case we assume the walls to be orthogonal to the x^1 -axis, placed at $x^1 = 0$ and $x^1 = L$, respectively. Let us consider the particular solution to the de Rham equations that, when $h_{\mu\nu}$ is switched off, describes a stationary electromagnetic field propagating along the x^1 direction with a linear polarization along the x^2 -axis. For the sake of simplicity we also consider a linearly polarized gravitational wave for which $H_{\times} = 0$ (see Sec. II). This solution is:

$$F_{02} = \sum_n -2 \mathcal{B}_n \sin(k_n x^1) \left[(1 - H f) \sin\left(k_n x^0 + \frac{H k_n}{2} f^{(-1)}\right) - \frac{H}{2 k_n} \dot{f} \cos\left(k_n x^0 + \frac{H k_n}{2} f^{(-1)}\right) \right] \quad (4.2)$$

$$F_{23} = \sum_n H \mathcal{B}_n \sin(k_n x^1) \left[f \sin\left(k_n x^0 + \frac{H k_n}{2} f^{(-1)}\right) + \frac{\dot{f}}{k_n} \cos\left(k_n x^0 + \frac{H k_n}{2} f^{(-1)}\right) \right] \quad (4.3)$$

$$F_{12} = \sum_n 2 \mathcal{B}_n \left(1 - \frac{H}{2} f\right) \cos(k_n x^1) \cos\left(k_n x^0 + \frac{H k_n}{2} f^{(-1)}\right) \quad (4.4)$$

where $k_n = \frac{n\pi}{L}$ and n is an integer. We notice that the solution is still stationary in the x^1 -direction. One can see that the gravitational wave induces a new component in the electromagnetic tensor (namely F_{23}). For a direct comparison to Ref. [12] we set $f(x^3 - x^0) = -\cos \chi(x^3 - x^0)$, and $\mathcal{B}_n = \mathcal{B}_{\bar{n}} \delta_{n\bar{n}}$. Performing a formal expansion in H , the electromagnetic field components are found to be a sum of the flat space-time solution and a perturbation, that is:

$$F_{\mu\nu} = {}^{(0)}F_{\mu\nu} + {}^{(1)}F_{\mu\nu} \quad (4.5)$$

where:

$${}^{(0)}F_{02} = -2 \mathcal{B}_{\bar{n}} \sin(k_{\bar{n}} x^1) \sin(k_{\bar{n}} x^0) \quad (4.6)$$

$${}^{(0)}F_{23} = 0 \quad (4.7)$$

$${}^{(0)}F_{12} = 2 \mathcal{B}_{\bar{n}} \cos(k_{\bar{n}} x^1) \cos(k_{\bar{n}} x^0) \quad (4.8)$$

are the “unperturbed” solutions, while

$$^{(1)}F_{02} = -2H\mathcal{B}_{\bar{n}} \sin(k_{\bar{n}}x^1) \left[\cos\chi(x^3 - x^0) \sin(k_{\bar{n}}x^1) - \frac{1}{2} \left(\frac{k_{\bar{n}}}{\chi} + \frac{\chi}{k_{\bar{n}}} \right) \sin\chi(x^3 - x^0) \cos(k_{\bar{n}}x^0) \right] \quad (4.9)$$

$$^{(1)}F_{23} = -H\mathcal{B}_{\bar{n}} \sin(k_{\bar{n}}x^1) \left[\cos\chi(x^3 - x^0) \sin(k_{\bar{n}}x^0) - \frac{\chi}{k_{\bar{n}}} \sin\chi(z - t) \cos(k_{\bar{n}}x^0) \right] \quad (4.10)$$

$$^{(1)}F_{12} = H\mathcal{B}_{\bar{n}} \cos(k_{\bar{n}}x^1) \left[\cos\chi(x^3 - x^0) \cos(k_{\bar{n}}x^1) + \frac{k_{\bar{n}}}{\chi} \sin\chi(z - t) \sin(k_{\bar{n}}x^0) \right] \quad (4.11)$$

would represent the perturbation due to the gravitational wave. Actually from the above expressions one immediately achieves that the formal expansion of the exact solution is meaningless when condition (4.1) does not hold true. In fact, in this case the “perturbation” terms are even greater than the “unperturbed” ones. On the other hand if Eq. (4.1) does hold, the splitting approach [12,20] yields a result that coincides with the formal expansion in H of our solution, but in a rotated reference frame.

It is important to notice that the solution expressed by Eqs. (4.2)–(4.4) can be cast in the form

$$A = \sum_n a_n(x^0, x^3 - x^0) A_n(\mathbf{x}), \quad (4.12)$$

where A stands for any electromagnetic tensor component, and A_n are the same as in a flat space–time. The effect of the gravitational wave results in a modulation of the a_n coefficients, and not in a creation of any new mode $A_{n'}$ ($n' \neq n$) even if $\chi = |k_n \pm k_{n'}|$.

B. Dissipative effects

As a second case of application we consider the walls to be orthogonal to the bisector to the x^1 – x^3 plane, placed in $x^1 = -x^3$ and $x^1 = \sqrt{2}L - x^3$, respectively. In a similar way to what we have done in the previous case, we consider a solution of the de Rham equation describing, when $h_{\mu\nu}$ vanishes, a stationary electromagnetic field propagating along the $x^1 = x^3$ direction with a linear polarization along the x^2 –axis. We set $h_\times = 0$ again. One has:

$$F_{02} = \sum_n -2\mathcal{B}_n \sin \left[\frac{k_n}{\sqrt{2}} \left(x^1 + x^3 - \frac{Hf^{(-1)}}{2} \right) \right] \left[(1 - Hf) \sin \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) - \frac{H}{2k_n} \dot{f} \cos \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) \right] \quad (4.13)$$

$$+ \frac{\mathcal{B}_n H f}{\sqrt{2}} \cos \left[\frac{k_n}{\sqrt{2}} \left(x^1 + x^3 - \frac{Hf^{(-1)}}{2} \right) \right] \cos \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) \\ F_{23} = \sum_n -\sqrt{2}\mathcal{B}_n (1 - Hf) \cos \left[\frac{k_n}{\sqrt{2}} \left(x^1 + x^3 - \frac{Hf^{(-1)}}{2} \right) \right] \cos \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) \quad (4.14)$$

$$+ \mathcal{B}_n H \sin \left[\frac{k_n}{\sqrt{2}} \left(x^1 + x^3 - \frac{Hf^{(-1)}}{2} \right) \right] \left[f \sin \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) + \frac{\dot{f}}{k_n} \cos \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right) \right] \\ F_{12} = \sum_n \sqrt{2}\mathcal{B}_n \left(1 - \frac{Hf}{2} \right) \cos \left[\frac{k_n}{\sqrt{2}} \left(x^1 + x^3 - \frac{Hf^{(-1)}}{2} \right) \right] \cos \left(k_n x^0 + \frac{k_n Hf^{(-1)}}{2} \right). \quad (4.15)$$

As in the previous case, the gravitational wave does not create any new modes of oscillation $A_{n'}$ [see Eq. (4.12)].

The most interesting feature of the solution is that the parallel component to the surface of the electric field does not vanish on the walls any more. This results in a *gravitationally induced* ohmic power loss within the conductors which is to be added to usual losses (such as those due to thermal vibrations of the conducting planes) already existing in flat space–time and not depending on the orientation of the apparatus. Consequently if the former kind of loss is at least comparable to the latter one it would be possible, in principle, to infer the presence of a gravitational wave by measuring the time decay of a resonant cavity.

We now proceed to estimate the *gravitationally induced* power loss per unit surface when

$$\frac{Hk_n}{\chi} \gg H. \quad (4.16)$$

This assumption leads to interesting results. In order to study dissipative currents induced by the gravitational wave on the walls, we only need to retain the biggest part of the field; it is noticeable that all terms proportional to H are negligible when Eq. (4.16) is fulfilled. One gets ($\mathcal{B}_n = \delta_{n\bar{n}} \mathcal{B}_{\bar{n}}$):

$$F_{02} = -2\mathcal{B}_{\bar{n}} \sin\left(\Psi - \frac{k_{\bar{n}} H f^{(-1)}}{2\sqrt{2}}\right) \sin\left(k_{\bar{n}} x^0 + \frac{k_{\bar{n}} H f^{(-1)}}{2}\right) \quad (4.17)$$

$$F_{23} = -\sqrt{2}\mathcal{B}_{\bar{n}} \cos\left(\Psi - \frac{k_{\bar{n}} H f^{(-1)}}{2\sqrt{2}}\right) \cos\left(k_{\bar{n}} x^0 + \frac{k_{\bar{n}} H f^{(-1)}}{2}\right) \quad (4.18)$$

$$F_{12} = -F_{23} \quad (4.19)$$

where $\Psi = 0$ if $x^1 = -x^3$ and $\Psi = k_{\bar{n}} L$ when $x^1 = -x^3 + \sqrt{2} L$. If we set $\mathcal{U} = \frac{dU}{da}$, where U is the electromagnetic energy and da stands for the surface element parallel to the walls, then the ohmic loss is given by [23]:

$$\frac{d\mathcal{U}}{dt} = 2 \frac{c^2 \mathbf{E}_{\parallel}^2}{8\pi \delta \omega} \quad (4.20)$$

where c is the speed of light (in the following we drop the convention $c = 1$, hence $x^0 = ct$ and $\Omega = \chi c$ stands for the gravitational wave frequency), \mathbf{E}_{\parallel} is the electric field component parallel to the surface, δ is the skin depth, and $\omega = c k_{\bar{n}}$ the electromagnetic frequency; factor 2 takes into account the fact that there are two walls. Therefore the equation describing the rate of energy loss per unit surface is:

$$-\frac{1}{\mathcal{U}} \frac{d\mathcal{U}}{dt} = \frac{2c^2}{L\delta\omega} \sin^2\left(\frac{k_{\bar{n}} H f^{(-1)}}{2\sqrt{2}}\right), \quad (4.21)$$

where an average over $\frac{2\pi}{\omega}$ has been performed. The solution to the above equation can be easily obtained assuming $\chi x^3 \sim 0$:

$$\mathcal{U}(t) = \mathcal{U}_0 \exp\left[-\int_0^t \frac{2c^2}{L\delta\omega} \sin^2\left(\frac{k_{\bar{n}} H f^{(-1)}(-t')}{2\sqrt{2}}\right) dt'\right]. \quad (4.22)$$

If we set $f(x) = \cos(\chi x)$ and make the further assumption (whose validity must be checked at the end) $\Omega t \ll 1$, Eq. (4.22) becomes:

$$\mathcal{U}(t) = \mathcal{U}_0 \exp\left[-\frac{2}{3} \frac{c^2}{L\delta\omega} \left(\frac{k_{\bar{n}} H}{2\sqrt{2}\chi}\right)^2 \Omega^2 t^3\right] \quad (4.23)$$

The time after which the energy falls by a factor $1/e$ (time decay) is:

$$\tau = \left[\frac{3}{2} \frac{L\delta\omega}{\Omega^2 c^2} \left(\frac{2\sqrt{2}\Omega}{\omega H}\right)^2\right]^{\frac{1}{3}}. \quad (4.24)$$

If the gravitational wave source considered is a binary pulsar as, for instance the PSR 1913 + 16 [24], then we find:

$$\tau = 5 (\delta L)^{\frac{1}{3}} 10^4 \text{ sec} \simeq 50 \text{ sec} \ll \frac{1}{\Omega} \sim 5 \cdot 10^3 \text{ sec} \quad (4.25)$$

where we have set $\omega = 10^{15} \text{ rad sec}^{-1}$ and $(\delta L)^{\frac{1}{3}} \simeq 10^{-3} \text{ cm}^{2/3}$ (e.g. [25]). Unfortunately this time lies many order of magnitude above the usual — non *gravitationally induced* — optical resonant cavity decay times.

However if we consider a coalescing compact binary [26] we get:

$$\Omega \tau - \frac{\sin(2\Omega\tau)}{2} = \frac{8\delta L \Omega^3}{c^2 \omega H^2} \quad (4.26)$$

where Eq. (4.22) has been used. Analysis of this equation, by taking $m_1 \sim m_2 \sim 3 m_{\odot}$, shows that the shortest decay time is obtained when the time to coalescing ranges within $2 \cdot 10^6 \text{ sec} \sim 23 \text{ d}$. In this case $\tau \sim 0.8 \text{ sec}$, to be compared with the decay times of the order 1–10 *msec* in a cavity. Therefore we see that this time is long, as compared to presently-measured decay times, but not so long to rule out the possibility to measure it in the future.

V. CONCLUSION

We have found the exact solution to the linearized Maxwell equations for a gravitational background described by a plane gravitational wave. We have not made any assumption over the shape or frequency of the gravitational wave. The only approximation used is the linearized Einstein theory, within which Eqs. (3.7) hold.

As an application we have evaluated the electromagnetic field bounced between two conducting walls. A comparison with known approximated results has been performed. We have found that, if Eq. (4.1) is met, the formal expansion in H of our solution reduces to the approximated one. Besides we have found out that the gravitational wave does not cause any new mode A_n [see Eq. (4.12)] to appear in the cavity. Furthermore we have evaluated a possible measurable effect of the interaction with low frequency gravitational waves. We have shown that in general an ohmic loss on the walls is produced; this results in a decrease of the stored electromagnetic energy.

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- [1] An excellent review of this topic is: P. R. Saulson, *Fundamentals of interferometric gravitational wave detectors* (World Scientific, Singapore, 1994) and references therein.
 - [2] N. L. Balazs, Phys. Rev. **110**, 236 (1958).
 - [3] J. Plebanski, Phys. Rev. **118**, 1396 (1960).
 - [4] C. V. Heer, Phys. Rev. **134**, A799 (1964).
 - [5] E. J. Post and A. Yildiz, Phys. Rev. Lett. **15**, 177 (1965).
 - [6] D. O. Muhleman and I. D. Johnston, Phys. Rev. Lett. **17**, 455 (1966).
 - [7] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational lenses* (Springer-Verlag, Berlin, 1992).
 - [8] C. Alcock, et al., Astrophys. J. **461**, 84 (1996); C. Alcock, et al., Astrophys. J. **486**, 697 (1997).
 - [9] D. M. Zipoy, Phys. Rev. **142**, 825 (1966).
 - [10] J. A. Lobo, Class. Quantum Grav. **9**, 1385 (1992).
 - [11] E. Montanari, Class. Quantum Grav. **15**, 2493 (1998).
 - [12] F. Cooperstock, Ann. Phys. **47**, 173 (1968).
 - [13] D. Boccaletti, V. de Sabbata, P. Fortini, C. Gualdi, Nuovo Cimento B **70**, 129 (1970).
 - [14] B. Bertotti, in *General Relativity and Cosmology, Course 47, International School of Physics "Enrico Fermi"*, edited by R. K. Sachs (Academic, New York, NY, 1971).
 - [15] V. B. Braginskii, L. P. Grishchuk, A. G. Doroshkevich, Ya. B. Zeldovich, I. D. Novikov, and M. V. Sazkin, Zh. Eksp. Teor. Fiz. **65**, 1729 (1973) [Sov. Phys. JETP **38**, 865 (1974)].
 - [16] H. Funke, Phys. Lett. A **67**, 439 (1978).
 - [17] J. M. Codina, J. Graells and C. Martin, Phys. Rev. D **21**, 2731 (1980).
 - [18] L. Baroni, P. Fortini and C. Gualdi, Ann. Phys. **162**, 49 (1985).
 - [19] V. B. Braginsky, N. S. Kardashev, G. Polnarev and I. D. Novikov, Nuovo Cimento B **105**, 1141 (1990).
 - [20] F. I. Cooperstock and V. Faraoni, Class. Quantum Grav. **10**, 1189 (1993).
 - [21] C. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
 - [22] P. G. Bergman, *Introduction to the theory of relativity*, (Dover, New York, 1975).
 - [23] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).
 - [24] R. A. Hulse and J. H. Taylor, Astrophys. J. Lett. **195**, 51 (1975).
 - [25] B. Rossi, *Optics* (Addison-Wesley, Reading, Massachusetts, 1957).
 - [26] R. V. Wagoner and C. Will, Astrophys. J. **210**, 764 (1976); E. Coccia and V. Fafone, Phys. Lett. A **213**, 16 (1996).